

Homework #3 (due May 10th)

Exercise 1. Let σ be the symbol defined by

$$\sigma(\xi) := e^{-|\xi|^2/2}, \quad \xi \in \mathbb{R}^n.$$

- (1) What is the kernel of the pseudodifferential operator T_σ ? Hint: Use the fact that $\widehat{\sigma}(\xi) = \sigma(\xi)$.
- (2) Show that $T_\sigma \varphi$ is real-analytic for any $\varphi \in C_c^\infty(\mathbb{R}^n)$. Hint: Use the kernel form of T_σ and compute the convergence radius of the Taylor series.
- (3) Conclude that the pseudodifferential operator T_σ does not map $C_c^\infty(\mathbb{R}^n)$ into itself.

Exercise 2. Fix $C > 0$ and $\chi \in C_c^\infty(\mathbb{R}^1)$. set $\sigma \in S^{-1}$ in \mathbb{R}^1 as

$$\sigma(x, \xi) = \frac{\chi(x)}{\xi + iC}.$$

Compute the kernel of T_σ .

Exercise 3. Use the identity

$$\mathcal{F}\{e^{\pm i|\cdot|^2/2}\}(\xi) = e^{\pm i\frac{\pi}{4}n} e^{\mp i|\xi|^2/2}$$

to derive

$$\mathcal{F}\{e^{\pm i\langle Q, \cdot \rangle/2}\}(\xi) = \frac{e^{\pm i\frac{\pi}{4}\text{sgn } Q}}{|\det Q|^{1/2}} e^{\mp i\langle Q^{-1}\xi, \xi \rangle/2}.$$

Exercise 4. (Optional) In (4.17), if we instead set

$$J(h, \chi f) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{\frac{\xi_j^2 h}{i2\alpha_j}} \cdot \widehat{\chi f}(\xi; 1/h) d\xi,$$

and later on expand J w.r.t. h at $h = 0$, will the computations following (4.17) still give the desired result? Explain the reason briefly.

Exercise 5. Assume $a \in C_c^\infty(\mathbb{R}^{2n})$ and denote a Lebesgue integral

$$I(y, \eta; \lambda) := (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i\lambda x \cdot \xi} a(x + y, \xi + \eta) dx d\xi.$$

- (1) fix y and η , and use Proposition 4.8 to find the asymptotic expansion of I w.r.t. λ as $\lambda \rightarrow +\infty$;
- (2) write down the first $1 + n$ terms (the leading term + the first order terms) of the asymptotic expansion.

Exercise 6. Assume symbol $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ and denote an oscillatory integral

$$I(y, \eta) := (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{ix \cdot \xi} a(x + y, \xi + \eta) dx d\xi.$$

- (1) is I well-defined? If it is, what would be the cutoff function? $\chi(\epsilon\xi)$, $\chi(\epsilon x)$, or $\chi(\epsilon x, \epsilon\xi)$?
- (2) use Proposition 4.3 to find the asymptotic expansion of I w.r.t. $\langle \eta \rangle$ as $|\eta| \rightarrow +\infty$;
- (3) write down the first $1 + n$ terms (the leading term + the first order terms) of the asymptotic expansion.

Hint 1: Perform the change of variable $\xi \rightarrow \langle \eta \rangle \xi$.

Hint 2: $x \cdot \xi = \langle Q(x, \xi), (x, \xi) \rangle$ with $Q = \begin{pmatrix} 0 & I_{n \times n} \\ I_{n \times n} & 0 \end{pmatrix}$, where (x, ξ) is treated as a vertical vector.