Homework #4 (due May 23rd)

Exercise 1. Use the identity

$$\mathcal{F}\{e^{\pm i\,|\cdot|^2/2}\}(\xi) = e^{\pm i\frac{\pi}{4}n} e^{\mp i|\xi|^2/2}$$

to derive

$$\mathcal{F}\{e^{\pm i\,\langle Q\cdot,\cdot\rangle/2}\}(\xi) = \frac{e^{\pm i\frac{\pi}{4}\operatorname{sgn} Q}}{|\det Q|^{1/2}} e^{\mp i\langle Q^{-1}\xi,\xi\rangle/2}.$$

Exercise 2. (Optional) In (4.17), if we instead set

$$J(h,\chi f) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{\frac{\xi_j^2 h}{i2\alpha_j}} \cdot \widehat{\chi f}(\xi; 1/h)\,\mathrm{d}\xi,$$

and later on expand J w.r.t. h at $h = 0$, will the computations following (4.17) still give the desired result? Explain the reason briefly.

Exercise 3. Assume $a \in C_c^\infty(\mathbb{R}^{2n})$ and denote a Lebesgue integral

$$I(y,\eta;\lambda) := (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i\lambda x\cdot\xi} a(x+y,\xi+\eta)\,\mathrm{d}x\,\mathrm{d}\xi.$$

(1) fix y and η, and use Proposition 4.8 to find the asymptotic expansion of I w.r.t. λ as $\lambda \to +\infty$;
(2) write down the first $1+n$ terms (the leading term + the first order terms) of the asymptotic expansion.

Exercise 4. Assume symbol $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ and denote an oscillatory integral

$$I(y,\eta) := (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{ix\cdot\xi} a(x+y,\xi+\eta)\,\mathrm{d}x\,\mathrm{d}\xi.$$

(1) is I well-defined? If it is, what would be the cutoff function? $\chi(\epsilon\xi)$, $\chi(\epsilon x)$, or $\chi(\epsilon x, \epsilon\xi)$?
(2) use Proposition 4.3 to find the asymptotic expansion of I w.r.t. $\langle\eta\rangle$ as $|\eta| \to +\infty$;
(3) write down the first $1+n$ terms (the leading term + the first order terms) of the asymptotic expansion.

Hint 1: Perform the change of variable $\xi \to \langle\eta\rangle\xi$.

Hint 2: $x\cdot\xi = \langle Q(x,\xi),(x,\xi)\rangle$ with $Q = \begin{pmatrix} 0 & I_{n\times n} \\ I_{n\times n} & 0 \end{pmatrix}$, where (x,ξ) is treated as a vertical vector.