Homework #3 (due April 25^{th})

Exercise 1. Prove $\mathscr{S}(\mathbb{R}^n) \subset S^{-\infty}$, namely, $\forall \varphi \in \mathscr{S}(\mathbb{R}^n)$, $\varphi(\xi) \in S^{-\infty}$.

Exercise 2. Let $m \in \mathbb{R}$. Let ψ be a smooth and uniformly bounded function in $\mathbb{R}^n$. Shows that $\sigma(x,\xi) = \psi(x)\langle\xi\rangle^m$ is a symbol of order m.

Exercise 3. Prove that the Dirac delta function $\delta \in H^{-s}(\mathbb{R}^n)$ whenever $s > n/2$.

Exercise 4. Let σ be the symbol defined by

$$\sigma(\xi) := e^{-|\xi|^2/2}, \quad \xi \in \mathbb{R}^n.$$

(1) What is the kernel of the pseudodifferential operator T_σ? Hint: Use the fact that $\widehat{\sigma}(\xi) = \sigma(\xi)$.
(2) Show that $T_\sigma\varphi$ is real-analytic for any $\varphi \in C_c^\infty(\mathbb{R}^n)$. Hint: Use the kernel form of T_σ and compute the convergence radius of the Taylor series.
(3) Conclude that the pseudodifferential operator T_σ does not map $C_c^\infty(\mathbb{R}^n)$ into itself.

Exercise 5. Fix $C > 0$ and $\chi \in C_c^\infty(\mathbb{R}^1)$. set $\sigma \in S^{-1}$ in $\mathbb{R}^1$ as

$$\sigma(x,\xi) = \frac{\chi(x)}{\xi + iC}.$$

Compute the kernel of T_σ.